Some structural graph properties of the non-commuting graph of a class of finite Moufang loops

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Abstract

For any non-abelian group G the non-commuting graph of G, $\Gamma = \Gamma_G$ is a graph with vertex set $G \backslash Z(G)$, where distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. This graph is connected for a non-abelian finite group and has received some attention in existing literature. Similarly, the non-commuting graph of a finite Moufang loop has been defined by the second author of this paper. He has shown that this graph is connected (as for groups) and obtained some results related to the non-commuting graph of a finite non-commutative Moufang loop.

In this paper, we show that the multiple complete split-like graphs are perfect (but not chordal) and deduce that the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ is perfect but not chordal. Then, we show that the non-commuting graph of a non-abelian group G is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$ is $3-$split and then classify all Chein loops that their non-commuting graphs are $3-$split. Precisely, we show that for a non-abelian group G, the non-commuting graph of the Moufang loop $M(G, 2)$, is $3-$split if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n. Finally, we calculate the energy of generalized and multiple split-like graphs and discuss about the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$.

Keywords Loop theory, Finite Moufang loops, Chein loops, Non-commuting graph of a finite group, Perfect graphs, Chordal graphs, Split graphs.
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1 Introduction

Let Q be a set with one binary operation. Then it is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. A quasigroup Q is a loop if Q possesses a neutral element e, i.e., if $ex = xe = x$ holds for every $x \in Q$. Moufang loops are loops in which any of the (equivalent) Moufang identities,

$$\begin{array}{ll} ((xy)x)z = x(y(xz)), & (M1) \\ x(y(zy)) = ((xy)z)y, & (M2) \\ (xy)(zx) = x((yz)x), & (M3) \\ (xy)(zx) = (x(yz))x. & (M4) \end{array}$$

holds for every $x, y, z \in Q$. Commutator of x, y and the associator of x, y and z are defined by $[x, y] = x^{-1}y^{-1}xy$ and $[x, y, z] = ((xy)z)^{-1}(x(yz))$, respectively. We define the commutant (or Moufang center) $C(Q)$ of Q as $\{x \in Q \mid xy = yx, \quad \forall y \in Q\}$. The center $Z(Q)$ of a Moufang loop Q is the set of all elements of Q which commute and associate with all other elements of Q. A non-empty subset P of Q is called a subloop of Q if P is itself a loop under the binary operation of Q, in particular, if this operation is associative on P, then it is a subgroup of Q. A subloop N of a loop Q is said to be normal in Q if $xN = Nx$; $x(yN) = (xy)N$; $N(xy) = (Nx)y$; for every $x, y \in Q$. In Moufang loop Q, the subloops $Z(Q)$ and $C(Q)$ are normal subloops. For more details about the Moufang loops one may see [7, 14, 11]. In 1974, Chein introduced a class of non-associative Moufang loops $M(G, 2)$, so called Chein loops. For a group G and a new element $u, (u \notin G)$, $M(G, 2) = G \cup Gu$ such that the multiplication with the new binary operation $\circ$ is defined as follows:

$$\begin{cases} g \circ h = gh, & g, h \in G, \\ g \circ (hu) = (hg)u, & g \in G,\ hu \in Gu, \\ (gu) \circ h = (gh^{-1})u, & gu \in Gu,\ h \in G, \\ (gu) \circ (hu) = h^{-1}g, & gu, hu \in Gu. \end{cases}$$

Clearly, the Moufang loop $M(G, 2)$ is non-associative if and only if G is non-abelian, see [7]. In [2], the second author has investigated some probabilistic properties of $M(G, 2)$, such as its *commutativity degree*.

Here, we recall the following two lemmas in order to prove some results.

Lemma 1.1. *([3], Lemma 3.7) In every Moufang loop $M(G, 2)$, we have $\forall x, y \in G$:*

(i) $(gu) \circ h = h \circ (gu)$ *if and only if* $h^2 = 1$;

(ii) $(gu) \circ (hu) = (hu) \circ (gu)$ *if and only if* $(g^{-1}h)^2 = 1$.

Also, if $gh = hg$ *then* $(gu) \circ (hu) = (hu) \circ (gu)$ *if and only if* $g^2 = h^2$. □

Lemma 1.2. *([3], Lemma 3.10) Let G be non-abelian group and $M = M(G, 2)$. Then the following statements hold:*

(*i*) $N(M) = Z(G)$;

(*ii*) $C(M) \subseteq G$. *Precisely,* $C(M) = Z(M) = \{x \in Z(G) \mid x^2 = 1\}$. □

There are many papers on assigning a graph to a ring or a group in order to investigation of their algebraic properties. For any non-abelian group G the non-commuting graph of G, $\Gamma = \Gamma_G$ is a graph with vertex set $G \backslash Z(G)$, where distinct non-central elements x and y of G are joined by an edge if and only if $xy \neq yx$. This graph is connected with diameter 2 and girth 3 for a non-abelian finite group and has received some attention in existing literature. For instance, one may see [1, 9, 13, 15]. Similarly, the non-commuting graph of a finite Moufang loop has been defined by the second author of this paper in [3]. He has defined this graph as follows: *Let M be a Moufang loop, then $M \backslash C(M)$ as the vertex set of this graph and two vertices x and y joined by an edge whenever* $[x, y] \neq 1$. He has shown that this graph is connected (as for groups) and obtained some results related to the non-commuting graph of a finite non-commutative Moufang loop.

We will denote a complete graph with n vertices by K_n. All graphs considered in this paper are finite and simple and also don't have any loop or multiple edges. For a graph Γ, we denote its vertex and edge sets by $V(\Gamma)$ and $E(\Gamma)$, respectively. The complement of Γ is denoted by $\overline{\Gamma}$. A graph $\Gamma = (V, E)$, is called $k-$partite where $k > 1$, if it is possible to partition V into k subsets $V_1, V_2, \ldots, V_k$, such that every edge of E joins a vertex of V_i to a vertex of V_j, $i \neq j$. A clique in a graph Γ is an induced subgraph whose all vertices are pairwise adjacent. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$. A subset X of the vertices of Γ is called an independent set (or stable) if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(\Gamma)$. The vertex chromatic number of a graph Γ is denoted by $\chi(\Gamma)$ and it is the minimum k for which $k-$vertex coloring of a graph Γ such that no two adjacent vertices have the same color. For a subset S of $V(\Gamma)$, $N_\Gamma[S]$ is the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$ then S is said to be a dominating set of the vertices in Γ. The minimum size of a dominating set of the vertices in Γ is dominating number of Γ and denoted by $\gamma(\Gamma)$. A vertex cover of a graph Γ is a set $Q \subseteq V(\Gamma)$ such that contains at least one endpoint of every edge. The minimum size of a vertex cover is denoted by $\beta(\Gamma)$. Our other used notations about graphs are standard and for more details one may see [6].

There is a relation between $\alpha(\Gamma)$ and $\beta(\Gamma)$ as follows:

Lemma 1.3. *([6], p. 296) Let Γ be a graph. Then $\alpha(\Gamma) + \beta(\Gamma) = n(\Gamma)$, where $n(\Gamma)$ is the number of vertices of Γ.* □

A perfect graph Γ, is a graph in which for every induced subgraph its clique number is equal to its chromatic number. A graph Γ is called weakly perfect graph if $\omega(\Gamma) = \chi(\Gamma)$. So, all perfect graphs are weakly perfect. A chordal graph is one in which all cycles of order four or more have a chord, which is an edge that is not part of cycle but connects two vertices of the cycle. The class of Chordal graphs is a subset of the class of perfect graphs. For more information about these types of graphs, one may see [10, 12]. We have the following theorem about perfect graphs, called *strong perfect graph theorem* or *Berge's Theorem*.

Theorem 1.4. *([8], Theorem 1.2) A graph Γ is perfect if and only if neither itself nor its graph complement $\overline{\Gamma}$ has a chordless cycle of odd order.* □

A graph is called k-regular, if the vertices of the graph are of the same degree k and a strongly regular graph S with parameters (n, k, λ, μ) is a $k-$regular graph of order n such that each pair of adjacent vertices has λ common neighbors and each pair of non-adjacent vertices has in which μ common neighbors. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be undirected simple graphs. The union $\Gamma_1 \cup \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph $\Gamma = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The notation $n\Gamma$ is short for $\underbrace{\Gamma \cup \cdots \cup \Gamma}_{n-times}$.

The complete product $\Gamma_1 \nabla \Gamma_2$ of graph Γ_1 and Γ_2 is a graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining every vertex of Γ_1 to every vertex of Γ_2. For every $a, b, n \in N$, a complete split, or simply, a split graph, is the graph $\bar{K}_a \nabla K_b$ and denoted by CS_b^a. By a theorem of Földes and Hammer ([10], Theorem 6.3), a graph is (complete) split iff contains no induced subgraph isomorphic to $2K2$, C_4 or C_5. Also, an undirected graph is split if and only if its complement is split ([10], Theorem 6.1). Clearly, every split graph is chordal and so perfect, but the converses are not true. More generally, a multiple complete split-like graph is $\bar{K}_a \nabla (nK_b)$ and denoted by $MCS_{b,n}^a$. Specially, in this paper, for $n = 3$ we call $MCS_{b,3}^a$ as a $3-$split graph.

Throughout this paper, we suppose that G is a non-abelian group. By using the following theorem and lemma, we will classify $3-$split graphs.

Lemma 1.5. *([4], Lemmas 2.4 and 2.5) If $G \setminus \{1\} = I \nabla C$ is a special split partition and $I^* = I \cup \{1\}$, then the following statements hold:*

(*i*) *Each element of C is an involution.*

(*ii*) *I^* is a maximal abelian normal subgroup of G and G/I^* is an elementary abelian 2-group.*

(*iii*) *Each vertex in C is certainly adjacent to each vertex in I. In particular, if $x \in C$, then $deg(x) = |G| - 2$, where if $y \in I$, then $deg(y) = |C|$.*

(*iv*) *I^* has odd order.*

(*v*) *G is centerless group, that is $Z(G) = 1$.* □

Theorem 1.6. *([4], Theorem 2.3) The non-commuting graph of a group G, Γ_G, is a complete split graph if and only if G is isomorphic to a Frobenius group of order $2n$ (n is odd) whose Frobenius kernel is abelian of order n.* □

We generalize the above definitions as follows: The generalized complete split-like graph is $GCS_k^a = \bar{K}_a \nabla S$ such that S is a strongly regular graph with parameters (n, k, λ, μ). The multiple generalized complete split-like graph is $GMCS_{k,m}^a = \bar{K}_a \nabla (mS)$. Let Γ be a simple graph on n vertices. The laplacian matrix of Γ is $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is its adjacency matrix and $D(\Gamma) = (d_1, \ldots, d_n)$ is the diagonal matrix of the vertex degrees in Γ. For any graph Γ, the energy of Γ is defined as $\xi(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of Γ. A spanning tree of a graph Γ is an induced subgraph of Γ, which is a tree and contains every vertex of Γ.

In this paper, we show that the multiple complete split-like graphs are perfect (but not chordal) and deduce that the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$ is perfect but not chordal. Then, we show that the non-commuting graph of a non-abelian group G is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$ is $3-$split and then classify all Chein loops that their non-commuting graphs are $3-$split. Precisely, we show that for a non-abelian group G, the non-commuting graph of the Moufang loop $M(G, 2)$, is $3-$split if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n. Finally, we calculate the energy of generalized and multiple split-like graphs and discuss about the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$. We recall the following proposition and theorems in order to provide some tools to these purposes.

Proposition 1.7. *([5], Theorem 4.11) Let Γ be a graph with n vertices and let L be the laplacian of Γ. Then the number of spanning trees of Γ is $\frac{1}{n^2} det(L + J)$ such that J denotes the all-one matrix.* □

Theorem 1.8. *([5], p. 3: Schur complement) Let A be a $n \times n$ matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are non-singular square matrices. Then the inverse of A, A^{-1} can be calculated by the following formula:*

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} (A/A_{11})^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} (A/A_{11})^{-1} \\ -(A/A_{11})^{-1} A_{21} A_{11} & (A/A_{11})^{-1} \end{bmatrix},$$

where

$$A/A_{11} = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

and

$$detA = detA_1 \times det(A_{22} - A_{21}A_{11}^{-1}A_{12}),$$

such that $detA$ is the determinant of A. □

Theorem 1.9. *([5], Theorems 6.2 and 6.22) Let S be a strongly regular graph with parameters (n, k, λ, μ). Then the adjacency matrix of S has exactly three distinct eigenvalues:*

(i) *k whose multiplicity is 1;*

(ii) $\frac{1}{2}((\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)})$,
whose multiplicity is: $\frac{1}{2}((n-1)-\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}})$;

(iii) $\frac{1}{2}((\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)})$,
whose multiplicity is: $\frac{1}{2}((n-1)+\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}})$. □

Theorem 1.10. *([12], Theorem 1) For $i = 1, 2$, let Γ_i be r_i-regular graphs with n_i vertices. Then the characteristic polynomial of the complete product of these two graphs is as follows:*

$$P_{\Gamma_1 \nabla \Gamma_2}(\lambda) = \frac{P_{\Gamma_1}(\lambda)P_{\Gamma_2}(\lambda)}{(\lambda-r_1)(\lambda-r_2)}[(\lambda-r_1)(\lambda-r_2)-n_1n_2].$$

□

2 Some basic graph properties of the Moufang loop $M(D_{2n}, 2)$

Let D_{2n} denote the dihedral group of order $2n$, which has the following presentation:

$$D_{2n} = \left\langle a, b | \quad a^n = b^2 = (ab)^2 = 1 \right\rangle.$$

From now on, we assume that $n \geq 3$. In this section, we want to study the non-commuting graph of the Moufang loops $M(D_{2n}, 2)$, simply denoted by Γ. We will use the following lemma in the sequel.

Lemma 2.1. *([3], Lemma 4.4) Let $M = M(D_{2n}, 2)$ and $\Gamma = \Gamma_M$ be its non-commuting graph.*

(a) *If n is odd then:*

(i) *$C(M) = 1$ and $|V(\Gamma)| = 4n - 1$;*

(*ii*) $\forall x \in M$,

$$|C_M(x)| = \begin{cases} 4n, & x = 1, \\ n, & x = a^i \quad (1 \leq i \leq n-1), \\ 2n+2, & x \neq a^i; \end{cases}$$

(*iii*) $\forall x \in M \setminus C(M)$,

$$deg(x) = \begin{cases} 3n, & x = a^i \quad (1 \leq i \leq n-1), \\ 2n-2, & x \neq a^i; \end{cases}$$

(*iv*) $\Sigma_{x \in M} |C_M(x)| = 7n^2 + 9n$,
$\Sigma_{x \in V(\Gamma)} deg(x) = 9n(n-1)$, $\quad |E(\Gamma)| = \frac{9n(n-1)}{2}$.

(*b*) *If* n *is even then:*

(*i*) $C(M) = \{1, a^{\frac{n}{2}}\}$ *and* $|V(\Gamma)| = 4n - 2$;

(*ii*) $\forall x \in M$,

$$|C_M(x)| = \begin{cases} 4n, & x = 1, a^{\frac{n}{2}}, \\ n, & x = a^i \quad (1 \leq i \leq n-1, i \neq \frac{n}{2}), \\ 2n+4, & x \neq a^i; \end{cases}$$

(*iii*) $\forall x \in M \setminus C(M)$,

$$deg(x) = \begin{cases} 3n, & x = a^i \quad (1 \leq i \leq n-1), \\ 2n-4, & x \neq a^i; \end{cases}$$

(*iv*) $\Sigma_{x \in M} |C_M(x)| = 7n^2 + 18n$,
$\Sigma_{x \in V(\Gamma)} deg(x) = 9n(n-2)$, $\quad |E(\Gamma)| = \frac{9n(n-2)}{2}$.

$\square$

The following lemma determines the structure of the non-commuting graph of the Moufang loop $M = M(D_{2n}, 2)$.

Lemma 2.2. *Let* $M = M(D_{2n}, 2)$ *and* $\Gamma = \Gamma_M$ *be its non-commuting graph.*

(*a*) *If* n *is odd then* $\Gamma_M \cong \bar{K}_{n-1} \nabla S$, *such that* S *is a strongly regular graph with parameters* $(3n, n-1, n-2, 0)$.

(*b*) *If* n *is even then* $\Gamma_M \cong \bar{K}_{n-2} \nabla 3S$, *such that* S *is a strongly regular graph with parameters* $(n, n-2, n-3, n-2)$.

Proof. a) By lemma 2.1 and the definition of the non-commuting graph, for every odd integer n, we can partition the vertices of Γ into four sets, as follows:

$$\begin{array}{llll} t_1 & = \{a, a^2, \ldots, a^{n-1}\}, & t_2 & = \{b, ab, \ldots, a^{n-1}b\}, \\ t_3 & = \{u, au, \ldots, a^{n-1}u\}, & t_4 & = \{bu, abu, \ldots, a^{n-1}bu\}. \end{array}$$

For every $0 \leq i, j \leq n-1$, since $a^i a^j = a^j a^i$, t_1 is an independent set and from the relations $a^i \circ (a^j b) \neq (a^j b) \circ a^i$, $a^i \circ (a^j u) \neq (a^j u)^i$ and $a^i \circ (a^j bu) \neq (a^j bu) \circ a^i$, we find that all vertices of t_1 are adjacent to all vertices of each of the sets t_2, t_3 and t_4. Also, by the relations $(a^i b) \circ (a^j b) \neq (a^j b) \circ (a^i b)$, the induced subgraph $[t_2]$ of Γ, is a clique. Similarly, we can show that the induced subgraph $[t_3]$ and $[t_4]$ of Γ, are cliques. Hence, $\Gamma \cong \bar{K}_{n-1} \nabla 3K_n$ and the graph Γ is $3-$split and $3K_n \cong S$, where S is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$.

b) Let n be an even integer. Again, we can partition the vertices of Γ into four sets, as follows:

$$\begin{array}{llll} t_1 & = \{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^{n-1}\}, & t_2 & = \{b, ab, \ldots, a^{n-1}b\}, \\ t_3 & = \{u, au, \ldots, a^{n-1}u\}, & t_4 & = \{bu, abu, \ldots, a^{n-1}bu\}. \end{array}$$

Since each pair of elements of t_1 commute, so the induced subgraph $[t_1]$ is an independent set, that means $[t_1] \cong \bar{K}_{n-2}$. Also, every element in M commutes with its inverse and since, $\forall x \in t_i, (i = 2, 3, 4)$, its inverse x^{-1} belongs to t_i. Therefore, every element of $t_i, (i = 2, 3, 4)$ is adjacent to each vertex in t_i, $i = 2, 3, 4$, except its inverse. Also any two elements x, y in $t_i, (i = 2, 3, 4)$ commute if and only if $|i - j| = \frac{n}{2}$, where $x = a^i u$ or $a^i b$, $a^i bu$ and $y = a^j u$ or $a^j b$, $a^j bu$. Then $[t_i] \cong S$, where S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Finally, for every $2 \leq i, j \leq 4$ there is no edge of Γ such that joins a vertex of t_i to a vertex of t_j, $i \neq j$, but each vertex in t_1 joins to each vertex in $t_i, (i = 2, 3, 4)$. Therefore, $\Gamma_M \cong \bar{K}_{n-2} \nabla 3S$. $\square$

In the following theorem, we derive some important graph properties of $\Gamma_{M(D_{2n},2)}$.

Theorem 2.3. *Let $M = M(D_{2n}, 2)$ and $\Gamma = \Gamma_M$ be its non-commuting graph.*

(*a*) *If n is odd then:*

$$\begin{array}{lll} \omega(\Gamma) = n+1, & \chi(\Gamma) = n+1, & \\ \alpha(\Gamma) = n-1, & \beta(\Gamma) = 3n, & \gamma(\Gamma) = 2. \end{array}$$

(*b*) *If n is even then:*

$$\omega(\Gamma) = \frac{n}{2} + 1, \qquad \chi(\Gamma) = \frac{n}{2} + 1,$$

$$\alpha(\Gamma) = \begin{cases} 6, & (n = 6) \\ n-2, & (n \geq 8) \end{cases}, \quad \beta(\Gamma) = \begin{cases} 16, & (n = 6) \\ 3n, & (n \geq 8) \end{cases}, \quad \gamma(\Gamma) = 2.$$

Proof. a) By lemma 2.2, the non-commuting graph of $M(D_{2n}, 2)$ is a generalized complete split-like graph for any odd integer n. Then $\Gamma \cong \bar{K}_{n-1}\nabla S$ in which S is a strongly regular graph with parameters $(3n, n-1, n-2, 0)$, where $V(\bar{K}_{n-1}) = \{a, a^2, \ldots, a^{n-1}\}$ and $S \cong 3K_n$. So this graph is $3-$split. By the structure of Γ, since every vertex of each copy of K_n is joined to the every vertex of $\bar{K}_{n-1}$, so we have the complete product $K_n\nabla[a^i]$, where $a^i \in \bar{K}_{n-1}$, $1 \leq i \leq n-1$. Also, $K_n\nabla[a^i]$ is the largest clique in Γ. So, $\omega(\Gamma) = n+1$. We need n distinct colors for coloring any K_n and only one color for coloring $\bar{K}_{n-1}$ which is distinct with the previous ones. So, $\chi(\Gamma) = n+1$. The set of vertices of $\bar{K}_{n-1}$ is the largest independent set, so $\alpha(\Gamma) = n-1$. By lemma 1.3, we have $\beta(\Gamma) = 4n-1-(n-1) = 3n$. Clearly, the set of vertices of $3K_n$ has the minimum size of a vertex cover. Any vertex of $\bar{K}_{n-1}$ is dominating all of vertices of S and any vertex of S is dominating all of vertices in $\bar{K}_{n-1}$. Thus $\gamma(\Gamma) = 2$.

b) By lemma 2.2, the non-commuting graph of $M(D_{2n}, 2)$, for every even integer n, is a multiple generalized complete split-like graph as $\Gamma = \bar{K}_{n-2}\nabla 3S$, such that S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$ and $\bar{K}_{n-2}$ is an independent set as follows:

$$V(\bar{K}_{n-2}) = \{a, a^2, \ldots, a^{\frac{n}{2}-1}, a^{\frac{n}{2}+1}, \ldots, a^n - 1\}.$$

In order to find the clique number, we may choose one vertex of $\bar{K}_{n-2}$ and the other vertices from only one copy of S's. By definition, every vertex is not joined to its inverse, so we can choose $\frac{n}{2}$ vertices of S and hence, $\omega(\Gamma) = \frac{n}{2}+1$. The color of every vertex in S is co-color with its inverse. Therefore, the chromatic number of S is equal to $\frac{n}{2}$ and so the maximum color number for all the vertices of $3S$ is equal to $\frac{n}{2}$. By only one color distinct from $\frac{n}{2}-$color in $3S$, we can color $\bar{K}_{n-2}$. So, $\chi(\Gamma) = \frac{n}{2} + 1$. For $n = 6$, $\bar{K}_{n-2}$ have four independent vertices, but with two non-adjacent vertices chosen from any of the copies of S, we get 6 independent vertices. Therefore, in this case $\alpha(\Gamma) = 6$. Now, for $n \geq 8$, the set $\bar{K}_{n-2}$ is the largest independent set and so, $\alpha(\Gamma) = n-2$. By using lemma 1.3, we have $\beta(\Gamma) = n(\Gamma) - \alpha(\Gamma)$. Hence, if $n = 6$ then $\beta(\Gamma) = 16$, else if $n \geq 8$ then $\beta(\Gamma) = 4n-2-(n-2) = 3n$. By choosing any vertex in $\bar{K}_{n-2}$ and the other in one of the copies of S, the domination set of Γ will be determined. Hence, $\gamma(\Gamma) = 2$. □

3 About perfectness and splitness of the non-commuting graph of a Moufang loop

In this section, first we show that the multiple complete split-like graphs are perfect and then characterize all Chein loops that their non-commuting graphs are $3-$split-like.

Theorem 3.1. *Every multiple complete split-like graph $MCS^a_{b,n} \cong \bar{K}_a \nabla (nK_b)$, $(n \geq 2)$ is perfect, but not chordal. Morever, every complete split graph $CS^a_{b,n} \cong \bar{K}_a \nabla K_b$, is perfect and also chordal.*

Proof. Let $\Gamma \cong \bar{K}_a \nabla (nK_b)$ and C be an odd cycle. If all vertices of C lie in only one copy of K_b's, clearly this cycle have a chord. Also, if some vertices of C lie in more than one copy of K_b's, then since in this case C have some vertices of $\bar{K}_a$ and also these vertices in $\bar{K}_a$ are adjacent to each vertex of K_b, therefore, the cycle have a chord. In addition, the complement graph, $\overline{\Gamma}$, is a disconnected graph of the form $K_a \cup S$ such that S is strongly regular graph with parameters $(nb, (n-1)b, (n-2)b, (n-1)b)$ or $S \cong T_{nb,b}$, which is a complete $n-$partite graph with nb vertices and hence, each part has b vertices. Clearly, any cycle in K_a have a chord. If C be an odd cycle in S, then by the structure of S, there is an intersection of C with more than three sections of S and these vertices are adjacent to any of the vertices in other sections and so, C have a chord. If C have an instruction with only two sections of S, then the induced subgraph of these sections will be a bipartite graph such that there is no any odd cycle in it. Now, by strong perfect graph theorem (theorem 1.4) Γ is a perfect graph. Let $\Gamma \cong \bar{K}_a \nabla (nK_b)$ and x_1, $x_2 \in \bar{K}_a$, $x_1 \neq x_2$. Take x_3 and x_4 from two distinct copies of K_b's. Now the induced subgraph of Γ generated by x_1, x_2, x_3 and x_4 is a cycle of length four without a chord. So, by definition, Γ is not chordal.

Similar to the proof of the first part, $CS^a_{b,n} \cong \bar{K}_a \nabla K_b$ is perfect, but there is not any cycle of length four or more without any chord and so this is a chordal graph. This completes the proof. □

Corollary 3.2. *The non-commuting graph of $M(D_{2n}, 2)$ is perfect but not chordal.*

Proof. Let $\Gamma = \Gamma(M(D_{2n}, 2))$, where n be an odd integer. Then by lemma 2.2*(a)*, $\Gamma \cong \bar{K}_{n-1} \nabla (3K_n)$ and by theorem 3.1, Γ is perfect but not chordal.

If n be an even integer then by lemma 2.2*(b)*, $\Gamma \cong \bar{K}_{n-2} \nabla 3S$ such that S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Assume that C is an odd cycle in Γ with length 5 or more, the length of the longest cycle without chord in each copy of S is equal to 4. Then there are some vertices of $\bar{K}_{n-2}$ in C and these vertices are adjacent to each vertex in $3S$. Therefore, C have a chord. On the other hand, $\overline{\Gamma} \cong K_{n-2} \cup (\frac{n}{2}K_2 \nabla \frac{n}{2}K_2 \nabla \frac{n}{2}K_2)$. Let C be a cycle in $\overline{\Gamma}$. Clearly, every cycle in K_{n-2} have a chord and if C be an odd cycle in $\frac{n}{2}K_2 \nabla \frac{n}{2}K_2 \nabla \frac{n}{2}K_2$, then C have an intersection with more than two parts of $\frac{n}{2}K_2$, where one of them have more than one vertex in C, and these vertices adjacent to all vertices of C in other parts and so, C have a chord and by theorem 1.4, Γ is perfect. The induced subgraph consist of any two vertices of $\bar{K}_{n-2}$ and two non-adjacent vertices of S is a cycle with length 4 without chord then Γ is not chordal. □

Remark 1. *The generalized multiple complete split-like graph $GMCS^a_k$ is not perfect. As a counterexample, let we have a generalized complete split-like graph $\Gamma \cong$*

$\bar{K}_a \nabla (nS)$ in which S is a Peterson graph. This graph is not perfect, since it has a cycle of length 5 without any chord. Recall that a Peterson graph is a strongly regular graph with parameters $(10, 3, 0, 1)$.

Theorem 3.3. *Let G be a non-abelian group. Then its non-commuting graph Γ_G, is split if and only if the non-commuting graph of the Moufang loop $M(G, 2)$, Γ_M, is $3-$split.*

Proof. Let Γ_M be $3-$split of the form $\Gamma_M = I \nabla 3C$, where I is an independent set and C is a complete graph. First we show that $Z(G) = C(M)$. By lemma 1.2, $C(M) \subseteq Z(G)$. Let $Z(G) \nsubseteq C(M)$. Then there exists $x \in Z(G)$ such that $x \notin C(M)$. Also, there exists $yu \in Gu$, where $x \circ (yu) \neq (yu) \circ x$, which yields $(yx)u \neq (yx^{-1})u$. Therefore, $x \neq x^{-1}$ and $x \in I$. So, every vertex y in each copy of C is adjacent to x and so $xy \neq yx$. But $x \in Z(G)$, i.e., for every $g \in G$, we have $xg = gx$. Hence $G \subseteq I$. Now, let $g \in G \setminus Z(G)$. So, there exist $t \in G$ such that $tg \neq gt$ but in this case $t, g \in I$ and this is a contradiction, since I is an independent set. So, $G = Z(G)$ and this contradicts with non-abelianity of G. Thus $Z(G) = C(M)$. Obviously, every element of $3C$ is an involution. Let $x \in 3C$ and $x \neq x^{-1}$. Since each element of Gu has order 2 then $x \in G$. Put $3C = C_1 \cup C_2 \cup C_3$, where each C_i is equal to a copy of C, $(1 \leq i \leq 3)$. Without loss of generality, let $x \in C_1$ and $x^{-1} \in C_2$ (note that $xx^{-1} = x^{-1}x$). Let $y \in G \backslash Z(G)$ and $y \notin \langle x \rangle$. Then since every element of G which commutes with x, also commutes with x^{-1}, so if $y \in C_1$ then $xy \neq yx$ and therefore $x^{-1}y \neq yx^{-1}$, but $x^{-1} \in C_2$ and this is a contradiction. Similarly, the case $y \in C_2$ will reach to a contradiction. So, $y \in I$ or $y \in C_3$. Now, consider the element xy. By the same reason as above, we have $xy \in I$ or $xy \in C_3$. Trivially, $xy \neq x, x^{-1}$. We have four cases as below:

Case 1. Let $y, xy \in I$. Then $y(xy) = (xy)y \Rightarrow yx = xy$, which is a contradiction, since y is adjacent to every element of C_1.

Case 2. Let $y \in I$ and $xy \in C_3$. Then $x \in C_1 \Rightarrow x(xy) = (xy)x$, $(x, y \in G) \Rightarrow xy = yx$ and we have the same contradiction as in case 1.

Case 3. Let $y \in C_3$ and $xy \in I$. Then $(xy)y \neq y(xy) \Rightarrow xy \neq yx$, which is also a contradiction since $y \in C_3$ and $x \in C_1$.

Case 4. Let $y, xy \in C_3$. Then we have $y(xy) \neq (xy)y \Rightarrow xy \neq yx$ and we obtain a similar contradiction as in case 3.

Therefore, every element of $3C$ has order 2. On the other hand, Γ_G is always connected and it is the induced subgraph of Γ_M. Therefore, $\Gamma_G \cong K_m$, $(K_m \subseteq C)$ or $\Gamma_G \cong I' \nabla nC'$ such that $I' \subseteq I$, $C' \subseteq C$ and $nC' = \cup_{i=1}^{n} C_i$, where $1 \leq n \leq 3$, and each C_i is a subset of one copy of C's. If $\Gamma_G \cong K_m$, then the order of every element of G will be equal to 2, so G must be abelian, which is absurd. Therefore, we get, $\Gamma_G \cong I' \nabla nC'$. If $n = 1$ then Γ_G is split. Suppose that $1 \neq x, y \in G$, $x \in C_1$

and $y \in C_2$, then $xy = yx$ and there exists $z \in I'$ where $yz \neq zy$ and $xz \neq zx$. So, $xy \in G$. If $xy \in I'$, then $x(xy) \neq (xy)x$ and so, $x^2y \neq x(yx)$. Therefore, $x^2y \neq x(xy)$ and this is a contradiction. If $xy \in C_1$ then $x(yx) \neq (xy)x$ and $x^2y \neq x^2y$, and it is a contradiction, and if $xy \in C_2$ then $y(xy) \neq (xy)y$ and $y^2x \neq y^2x$, and it is also a contradiction. Finally, let $xy \in C_3$. Now, $xu \in M(G, 2)$ then:

1) If $xu \in I$ or $xu \in C_1$, then $(xu) \circ x \neq x \circ (xu)$ and so $(xx^{-1})u \neq (xx)u$. Therefore, $u \neq x^2u$, this is a contradiction. So, every element of C in Γ_M is of order 2 therefore, $x^2 = 1$.

2) If $xu \in C_2$ then $(xu) \circ y \neq y \circ (xu)$ and so $(xy^{-1})u \neq (xy)u$. Thus $(xy)u \neq (xy)u$ and this is a contradiction.

3) If $xu \in C_3$ then $(xu) \circ (xy) \neq (xy) \circ (xu)$ and so $(x(xy)^{-1})u \neq (x(xy))u$ or $(x(y^{-1}x^{-1}))u \neq (x^2y)u$. So, $(x(yx))u \neq (x^2y)u$, or $(x(xy))u \neq yu$. Thus $(x^2y)u \neq yu$ and this is a contradiction.

Therefore, $\Gamma_G \cong I'\nabla C'$ and Γ_G is split.

Conversely, let Γ_G be split. Then $\Gamma_G \cong I\nabla C$. We show that Γ_M is $3-$split. By splitness of Γ_G and lemma 1.5, we have, $Z(G) = 1$ and $C(M) \subseteq Z(G)$. So, $C(M) = 1$. Let $V(I) = \{a_1, a_2, \ldots, a_k\}$ and $V(C) = \{b_1, b_2, \ldots, b_t\}$. Then, $V(\Gamma_M)$ includes $V(I)$, $V(C)$ and the set of vertices of the form, $V(Iu) = \{a_1u, a_2u, \ldots, a_ku\}$ and $V(Cu) = \{b_1u, b_2u, \ldots, b_tu\}$. To prove $3-$splitness Γ_M, we consider and stablish the following claims.

Claim 1. *The induced subgraph containing the vertices in $V(Iu)$ forms a clique.*

Let there exist two non-adjacent vertices a_iu and a_ju. So, $(a_iu) \circ (a_ju) = (a_ju) \circ (a_iu)$ and then $a_ia_j^{-1} = a_ja_i^{-1}$ or $(a_ia_j^{-1})^2 = 1$. Therefore, by lemma 1.5, $I^* = I \cup \{1\}$ is a maximal subgroup of odd order and there is not any element of even order. So, $a_ia_j^{-1} \in C$, where in this case $(a_ia_j^{-1})a_j \neq a_j(a_ia_j^{-1})$. Then $a_i \neq a_j(a_ia_j^{-1})$ and $a_j^{-1}a_i \neq a_ia_j^{-1}$ and this is a contradiction.

Claim 2. *The induced subgraph containing the vertices in $V(Cu)$ is a clique.*

Let there exist two vertices b_iu and b_ju such that are not adjacent. So, $(b_iu) \circ (b_ju) = (b_ju) \circ (b_iu)$. Therefore, $b_ib_j^{-1} = b_jb_i^{-1}$ and so $b_ib_j = b_jb_i$. Since, each element of C is an involution, this yields to a contradiction.

Claim 3. *There is no edge between $V(Iu)$ and $V(Cu)$.*

Let there exist two vertices a_iu and b_ju such that $(a_iu) \circ (b_ju) \neq (b_ju) \circ (a_iu)$ then $b_j^{-1}a_i \neq a_i^{-1}b_j$ and $b_ja_i \neq a_i^{-1}b_j$, therefore $(b_ja_i)^2 \neq 1$. On the other hand $b_ja_i \in G$. So, $b_ja_i \in I$ or $b_ja_i \in C$.

1) If $b_ja_i \in I$ then $(b_ja_i)a_i = a_i(b_ja_i)$ and $b_ja_i = a_ib_j$, which yields to a contradiction.

2) If $b_j a_i \in C$ then $(b_j a_i)^2 = 1$ and this is a contradiction. Therefore, any two elements of $V(Iu)$ and $V(Cu)$ are non-adjacent.

Claim 4. *There is no edge between $V(C)$ and $V(Cu)$.*

Let there exist two vertices b_i and $b_j u$ such that $b_i \circ (b_j u) \neq (b_j u) \circ b_i$. Then $(b_j b_i)u \neq (b_j b_i^{-1})u$, so, $(b_j b_i)u \neq (b_j b_i)u$ and this is a contradiction. Therefore any two elements of $V(C)$ and $V(Cu)$ are non-adjacent.

Claim 5. *There is no edge between $V(C)$ and $V(Iu)$.*

Let there exist two vertices b_i and $a_j u$ such that $b_i \circ (a_j u) \neq (a_j u) \circ b_i$. Then $(a_j b_i)u \neq (a_j b_i^{-1})u$ and $a_j b_i \neq a_j b_i$. This is a contradiction. Therefore, any two vertices in $V(C)$ and $V(Iu)$ are non-adjacent.

Claim 6. *Every vertex in $V(Iu)$ is adjacent to every vertex in $V(I)$.*

Let there exist two vertices a_i and $a_j u$ such that $a_i \circ (a_j u) = (a_j u) \circ a_i$. Then $(a_j a_i)u = (a_j a_i^{-1})u$ and $a_j a_i = a_j a_i^{-1}$. So, $a_i = a_i^{-1}$. Therefore, $a_i^2 = 1$ and this is a contradiction.

Claim 7. *Every vertex in $V(Cu)$ is adjacent to every vertex in $V(I)$.*

Let there exist two vertices $a_i \in I$ and $b_j u \in Cu$ such that $a_i \circ (b_j u) = (b_j u) \circ a_i$. Also, $(b_j a_i)u = (b_j a_i^{-1})u$ then $b_j a_i = b_j a_i^{-1}$ and $a_i = a_i^{-1}$, therefore $a_i^2 = 1$ and this is a contradiction.

Thus the non-commuting graph of $M(G, 2)$ is $3-$split, where the induced subgraphs containing the vertices of C and Cu and Iu are cliques and I is an independent set. □

Now, by using theorems 1.6 and 3.3, we can classify all $3-$split Chein loops:

Corollary 3.4. *Let G be a non-abelian group. Then the non-commuting graph of the Moufang loop $M(G, 2)$, is $3-$split if and only if G is isomorphic to a Frobenius group of order $2n$, n is odd, whose Frobenius kernel is abelian of order n.* □

4 About the energy and the number of spanning trees of generalized and multiple split-like graphs

In this section, we are going to calculate the energy of generalized complete and multiple split-like graphs and derive the energy and also the number of spanning trees in the case of the non-commuting graph of Chein loops of the form $M(D_{2n}, 2)$.

Theorem 4.1. *Let Γ be a generalized complete split-like graph, $\Gamma \cong \bar{K}_a \nabla (nK_b)$. Then $\varepsilon(\Gamma) = 2n(b-1)$.*

Proof. Let $P_{K_b}(\lambda)$ be the characteristic polynomial of K_b. Then,

$$P_{K_b}(\lambda) = (-1)^b(\lambda+1)^{b-1}(\lambda-b+1).$$

So,

$$P_{nK_b}(\lambda) = (-1)^{nb}(\lambda+1)^{n(b-1)}(\lambda-b+1)^n$$

and

$$P_{\bar{K}_a}(\lambda) = (-\lambda)^a.$$

By using theorem 1.10, we have:

$$P_{\Gamma}(\lambda) = (-1)^{nb+a}(\lambda+1)^{n(b-1)}(\lambda-b+1)^{n-1}\lambda^{a-1}(\lambda^2-(b-1)\lambda-nab)$$

and by definition of the energy of a graph, we get:

$$\varepsilon(\Gamma) = n(b-1)+(n-1)(b-1)+b-1.$$

Hence, $\varepsilon(\Gamma) = 2n(b-1)$. □

Corollary 4.2. *Let $G = D_{2n}$ and $M = M(G, 2)$. Then:*

- (i) *if n is an odd integer, then$\varepsilon(\Gamma_M) = 6(n-1)$;*
- (i) *if n is an even integer, then $\varepsilon(\Gamma_M) = 6(n-2)$.*

Moreover, in both cases, $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$.

Proof. i) Since, $\Gamma_M \cong \bar{K}_{n-1}\nabla 3K_n$, by theorem 4.1, $\varepsilon(\Gamma_M) = 6(n-1)$. We know that $\Gamma_G \cong \bar{K}_{n-1}\nabla K_n$ and by theorem 4.1, we have $\varepsilon(\Gamma_G) = 2(n-1)$. Thus $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$.
ii) Now, let n be an even integer. Then, by theorem 2.2, $\Gamma_M \cong \bar{K}_{n-2}\nabla 3S$, in which S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Thus, by theorem 1.9, the adjacency matrix of S has exactly three distinct eigenvalues: $\lambda_1 = n-2$, whose multiplicity is 1, $\lambda_2 = 0$, whose multiplicity is 1 and $\lambda_3 = -1$, whose multiplicity is $n-2$. Therefore,

$$P_S(\lambda) = (\lambda-n+2)(\lambda+1)^{n-2}\lambda.$$

So,

$$P_{3S}(\lambda) = (\lambda-n+2)^3(\lambda+1)^{3n-6}\lambda^3$$

and

$$P_{\bar{K}_{n-2}}(\lambda) = \lambda^{n-2}.$$

By theorem 1.10, we have:

$$P_{\Gamma_M}(\lambda) = (\lambda-n+2)^2(\lambda+1)^{3n-6}\lambda^{n-2}(\lambda^2+(2-n)\lambda-3n(n-2)).$$

Thus, $\varepsilon(\Gamma_M) = 6(n-2)$. We know that $\Gamma_G \cong \bar{K}_{n-2}\nabla S$, such that S is a strongly regular graph with parameters $(n, n-2, n-3, n-2)$. Therefore, by theorem 1.9,

$$P_{\Gamma_G}(\lambda) = (\lambda+1)^{n-2}\lambda^{n-2}(\lambda^2 + (2-n)\lambda - n(n-2)).$$

So, $\varepsilon(\Gamma_G) = 2(n-2)$. Thus $\varepsilon(\Gamma_G)$ divides $\varepsilon(\Gamma_M)$. □

Finally, in the following theorems, we count the number of spanning trees of the non-commuting graph Γ_M, where $M = M(D_{2n}, 2)$, for odd and even n, separately, and they lead us to an important result.

Theorem 4.3. *The number of spanning trees of the non-commuting graph Γ_M, where $M = M(D_{2n}, 2)$ and n is odd, is equal to:*

$$\kappa(\Gamma_M) = (2n-1)^{3n-3}(n-1)^2(3n)^{n-2}.$$

Proof. There are $4n-1$ vertices in this graph, such that they are in t_1, t_2, t_3, t_4. Each of t_i, $2 \leq i \leq 4$, have n vertices of degree $2n-2$ and t_1 have $n-1$ vertices of degrees $3n$. By the structure of graph Γ_M in 2.2, the matrix of vertex degree, namely D of this graph is equal to:

$$D = \begin{bmatrix} (2n-2)I_{3n} & 0_{3n(n-1)} \\ 0_{(n-1)3n} & (3n)I_{n-1} \end{bmatrix}$$

and the adjacent matrix of graph has the form:

$$A = \begin{bmatrix} (J_n - I_n)\bigotimes I_3 & J_{3n(n-1)} \\ J_{(n-1)3n} & 0_{n-1} \end{bmatrix},$$

where, $\bigotimes$ denotes the tensor product of matrices. Thus,

$$L = D - A = \begin{bmatrix} ((2n-1)I_n - J_n)\bigotimes I_3 & -J_{3n(n-1)} \\ -J_{(n-1)3n} & (3n)I_{n-1} \end{bmatrix}.$$

Now, to calculate $\det(L+J)$, we have

$$L + J = \begin{bmatrix} (2n-1)I_n & J_n & J_n & 0 \\ J_n & (2n-1)I_n & J_n & 0 \\ J_n & J_n & (2n-1)I_n & 0 \\ 0 & 0 & 0 & (3n)I_{n-1} + J_{n-1} \end{bmatrix}.$$

Also, in this case we have

$$\det(L+J) = \det B \times \det C, \tag{1}$$

where,

$$B = \begin{bmatrix} (2n-1)I_n & J_n & J_n \\ J_n & (2n-1)I_n & J_n \\ J_n & J_n & (2n-1)I_n \end{bmatrix}$$

and $C = (3n)I_{n-1} + J_{n-1}$. So,

$$\det C = (3n)^{n-2}(4n-1) \tag{2}$$

and

$$B = \begin{bmatrix} E & J_{(2n)n} \\ J_{n(2n)} & F \end{bmatrix},$$

where,

$$E = \begin{bmatrix} (2n-1)I_n & J_n \\ J_n & (2n-1)I_n \end{bmatrix}$$

and $F = (2n-1)I_n$. By theorem 1.8, we have

$$\det B = \det F \times \det(E - JF^{-1}J). \tag{3}$$

So, by using the following relations

$$\det F = (2n-1)^n, \qquad F^{-1} = \frac{1}{2n-1}I_n, \qquad JF^{-1}J = \frac{n}{2n-1}J_{2n}, \tag{4}$$

we have

$$E - JF^{-1}J = \frac{1}{2n-1}\begin{bmatrix} G & (n-1)J \\ (n-1)J & G \end{bmatrix},$$

where, $G = (2n-1)^2 I - nJ$ and

$$\det G = (2n-1)^{2n-2}(n-1)(3n-1), \quad G^{-1} = \frac{1}{(2n-1)^2}(I + \frac{n}{(n-1)(3n-1)}J). \tag{5}$$

Now,

$$\det(E - JF^{-1}J) = (\frac{1}{2n-1})^{2n}\det(G) \times \det(G - (n-1)^2 JG^{-1}J), \tag{6}$$

where,

$$(n-1)^2 JG^{-1}J = \frac{n(n-1)}{3n-1}J$$

and

$$G - (n-1)^2 JG^{-1}J = \frac{1}{3n-1}((\alpha-\beta)I + \beta J),$$

such that, $\alpha = (n-1)(2n-1)(6n-1)$ and $\beta = -2n(2n-1)$. So,

$$\det(G - (n-1)^2 JG^{-1}J) = (2n-1)^{2(n-1)}\frac{8n^3 - 14n^2 + 7n - 1}{3n-1}. \tag{7}$$

By using the relations 5, 6 and 7, we have

$$\det(E - JF^{-1}J) = (2n-1)^{2(n-2)}(n-1)(8n^3 - 14n^2 + 7n - 1) \tag{8}$$

and by replacing relations 4 and 8 in 3 we get

$$\det B = (2n-1)^{3n-4}(n-1)(8n^3 - 14n^2 + 7n - 1). \tag{9}$$

Now, by replacing relations 2 and 9 in 1, we get

$$\det(L+J) = (2n-1)^{3(n-1)}(n-1)^2(4n-1)^2(3n)^{n-2}.$$

By proposition 1.7, we have $\kappa = \frac{\det(L+J)}{(4n-1)^2}$. Therefore,

$$\kappa(\Gamma_M) = (2n-1)^{3(n-1)}(n-1)^2(3n)^{n-2}.$$

□

Theorem 4.4. *The number of spanning trees of the non-commuting graph Γ_M, where, $M = M(D_{2n}, 2)$ and n is even, is equal to:*

$$\kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}.$$

Proof. There are $4n-2$ vertices in this graph and they are in t_1, t_2, t_3, t_4. Each of t_i, $2 \le i \le 4$, have n vertices of degree $2n-4$ and t_1 have $n-2$ vertices of degree $3n$. By the structure of the graph Γ in 2.2, the matrix of the vertex degree namely D, of this graph is:

$$D = \begin{bmatrix} 2(n-2)I_{3n} & 0 \\ 0 & 3nI_{n-2} \end{bmatrix}$$

and the adjacent matrix of the graph has the form:

$$A = \begin{bmatrix} X_n & 0 & 0 & J \\ 0 & X_n & 0 & J \\ 0 & 0 & X_n & J \\ J & J & J & 0 \end{bmatrix}.$$

By lemma 2.2, each vertex in every $t_i \quad (2 \le i \le 4)$, is adjacent to the other vertices except its inverse element and itself and so,

$$X = \begin{bmatrix} J-I & J-I \\ J-I & J-I \end{bmatrix},$$

such that I and J are square matrices of order $\frac{n}{2}$ in X. So,

$$L = D - A = \begin{bmatrix} Y_n & 0 & 0 & -J \\ 0 & Y_n & 0 & -J \\ 0 & 0 & Y_n & -J \\ -J & -J & -J & 3nI_{n-2} \end{bmatrix},$$

such that,

$$Y = \begin{bmatrix} (2n-3)I - J & I - J \\ I - J & (2n-3)I - J \end{bmatrix}.$$

Hence,

$$L + J = \begin{bmatrix} Z & J & J & 0 \\ J & Z & J & 0 \\ J & J & Z & 0 \\ 0 & 0 & 0 & 3nI + J \end{bmatrix}.$$

We have

$$Z = Y + J = \begin{bmatrix} (2n-3)I & I \\ I & (2n-3)I \end{bmatrix},$$

in which the order of I is equal to $\frac{n}{2}$. Now we obtain

$$\det(L + J) = \det B \times \det C, \tag{10}$$

where $C = 3nI_{n-2} + J_{n-2}$ and

$$B = \begin{bmatrix} Z & J & J \\ J & Z & J \\ J & J & Z \end{bmatrix}.$$

Therefore,

$$\det C = 2(3n)^{n-3}(2n-1) \tag{11}$$

and by using theorem 1.8, we have

$$\det B = \det Z \times \det(E - JZ^{-1}J), \tag{12}$$

where,

$$E = \begin{bmatrix} Z & J \\ J & Z \end{bmatrix}$$

and

$$\det Z = (4(n-1)(n-2))^{\frac{n}{2}}. \tag{13}$$

Also,

$$Z^{-1} = \frac{1}{(2n-3)^2 - 1} \begin{bmatrix} (2n-3)I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & (2n-3)I_{\frac{n}{2}} \end{bmatrix}$$

and so, $JZ^{-1} = \frac{1}{2(n-1)} J_{2n \times n}$ and $JZ^{-1}J = \frac{n}{2(n-1)} J_{2n \times 2n}$. So,

$$E - JZ^{-1}J = \begin{bmatrix} G & H \\ H & G \end{bmatrix}, \tag{14}$$

such that, $H = \frac{n-2}{2(n-1)}J$ and

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where, $G_{11} = G_{22} = (2n-3)I - \frac{n}{2(n-1)}J$ and $G_{12} = G_{21} = I - \frac{n}{2(n-1)}J$.

By using elementary row or column operations in G we have

$$\begin{aligned} \det G &= \det(\tfrac{1}{2(n-1)} \begin{bmatrix} (n-1)(4n-6)I - nJ & 4(n-2)(1-n)I \\ 4(n-2)(1-n)I & 8(n-1)(n-2)I \end{bmatrix}) \\ &= (8(n-1)(n-2))^{\frac{n}{2}} \tfrac{1}{(2(n-1))^n} \det(2(n-1)^2 I - nJ). \end{aligned}$$

Since,

$$\det(2(n-1)^2 I - nJ) = 2^{\frac{n}{2}-2}(n-1)^{n-2}(n-2)(3n-2),$$

then

$$\det G = 2^{n-2}(n-1)^{\frac{n}{2}-2}(n-2)^{\frac{n}{2}+1}(3n-2). \tag{15}$$

By theorem 1.8, G^{-1} is as follows:

$$G^{-1} = \begin{bmatrix} G_{11}^{-1} + (G_{11}^{-1}G_{12})(G/G_{11})^{-1}(G_{12}G_{11}^{-1}) & -G_{11}^{-1}G_{12}(G/G_{11})^{-1} \\ -(G/G_{11})^{-1}G_{12}G_{11}^{-1} & (G/G_{11})^{-1} \end{bmatrix},$$

such that, $G/G_{11} = G_{11} - G_{12}G_{11}^{-1}G_{12}$. Therefore,

$$G_{11}^{-1} = \frac{1}{(n-1)(2n-3)}(\frac{1}{2}I + \frac{n}{(n-2)(7n-6)}J)$$

and $G_{12} = I - \frac{n}{2(n-1)}J$. Then:

$$G_{12}G_{11}^{-1}G_{12} = \frac{1}{(2n-3)}(2(n-1)I + \frac{n(2n^2-15n+14)}{(7n-6)}J)$$

and

$$G_{11} - G_{12}G_{11}^{-1}G_{12} = \frac{8(n-1)(n-2)}{2n-3}((n-1)I - \frac{2n}{7n-6}J).$$

Now, we have

$$G/G_{11} = \frac{8(n-1)(n-2)}{(2n-3)}((n-1)I - \frac{2n}{7n-6}J),$$

and

$$(G/G_{11})^{-1} = \frac{1}{8(n-1)^2(n-2)}((2n-3)I + \frac{2n}{3n-2}J).$$

Therefore,

$$G^{-1} = \frac{1}{4(n-1)(n-2)}(\begin{bmatrix}(2n-3)I & -I \\ -I & (2n-3)I\end{bmatrix} + \frac{2n}{3n-2}J).$$

Also, $HG^{-1}H = \frac{n(n-2)}{2(n-1)(3n-2)}J$ and

$$G - HG^{-1}H = \begin{bmatrix}(2n-3)I & I \\ I & (2n-3)I\end{bmatrix} - \frac{2n}{3n-2}J.$$

By using elementary row or column operations, we have

$$\det(G - HG^{-1}H) = \frac{2^n}{(3n-2)}(n-2)^{\frac{n}{2}+1}(n-1)^{\frac{n}{2}-1}(2n-1) \tag{16}$$

By relation 14, we get

$$\det(E - JZ^{-1}J) = \det G \times \det(G - HG^{-1}H).$$

Then, by relations 15 and 16, we have

$$\det(E - JZ^{-1}J) = 2^{2n-2}(n-1)^{n-3}(n-2)^{n+2}(2n-1). \tag{17}$$

Also, from relations 12, 13 and 17, we obtain

$$\det B = 2^{3n-2}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}(2n-1) \tag{18}$$

and by relations 10, 11 and 18, we have

$$\det(L + J) = 2^{3n-1}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}(2n-1)^2, \tag{19}$$

and from replacing 19 in $\kappa = \frac{\det(L+J)}{(4n-2)^2}$, we get

$$\kappa(\Gamma_M) = 2^{3n-3}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}.$$

□

Corollary 4.5. *Let $M = M(G, 2)$, where $G = D_{2n}$. Then $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$.*

Proof. By example 1 in [4], the non-commuting graph of $G = D_{2n}$, when in is odd, is a split graph and $\Gamma_G \cong I\nabla C$, where I is an independent set with $n-1$ vertices and $C \cong K_n$. So, the degree matrix of Γ_G has the form:

$$D = \begin{bmatrix}(2n-2)I_{n-1} & 0 \\ 0 & nI_n\end{bmatrix}$$

and the adjacency matrix of Γ_G is equal to:

$$A = \begin{bmatrix} J-I & J \\ J & 0 \end{bmatrix}.$$

So,

$$L = D - A = \begin{bmatrix} 2n-1)I - J & -J \\ -J & nI \end{bmatrix}$$

and

$$L + J = \begin{bmatrix} (2n-2)I & 0 \\ 0 & nI + J \end{bmatrix}.$$

Thus, $\det(L+J) = \det((2n-1)I) \times \det(nI+J)$ and this gives us:

$$\det(L+J) = (2n-1)^{n+1} n^{n-2}.$$

Therefore,

$$\kappa(\Gamma_G) = \frac{\det(L+J)}{(2n-1)^2} = (2n-1)^{n-1} n^{n-2}.$$

By theorem 4.3, $\kappa(\Gamma_M) = (2n-1)^{3(n-1)}(n-1)^2(3n)^{n-2}$. Hence, the proof is complete and $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$, where n is an odd integer.

Now, let n be an even integer. Then $\Gamma_G \cong \bar{K}_{n-2}\nabla S$, where S is a strongly regular graph with parameters $(n, n-2, n-4, n-2)$. Also, the degree matrix, D, of Γ_G is equal to:

$$D = \begin{bmatrix} (2n-4)I & 0 \\ 0 & nI \end{bmatrix}$$

and the adjacency matrix of Γ_G, namely A, has the form:

$$A = \begin{bmatrix} X & J \\ J & 0 \end{bmatrix},$$

where,

$$X = \begin{bmatrix} J-I & J-I \\ J-I & J-I \end{bmatrix},$$

in which, I and J are of order $\frac{n}{2}$. So,

$$L = D - A = \begin{bmatrix} Y & -J \\ -J & nI \end{bmatrix},$$

where,

$$Y = \begin{bmatrix} (2n-3)I - J & I - J \\ I - J & (2n-3)I - J \end{bmatrix}.$$

Hence,

$$L+J=\begin{bmatrix} Z & 0 \\ 0 & nI+J \end{bmatrix},$$

where,

$$Z=\begin{bmatrix} (2n-3)I & I \\ I & (2n-3)I \end{bmatrix}.$$

Since, $\det(L+J)=\det Z\times\det(nI+J)$, $\det Z=(4(n-1)(n-2))^{\frac{n}{2}}$ and $\det(nI+J)=n^{n-3}(2n-2)$, then

$$\det(L+J)=2^{n+1}n^{n-3}(n-1)^{\frac{n}{2}+1}(n-2)^{\frac{n}{2}}.$$

Therefore,

$$\kappa(\Gamma_G)=\frac{\det(L+J)}{(2n-2)^2}=2^{n-1}n^{n-3}(n-1)^{\frac{n}{2}-1}(n-2)^{\frac{n}{2}}.$$

Also, by theorem 4.4, we have

$$\kappa(\Gamma_M)=2^{3n-3}(3n)^{n-3}(n-1)^{\frac{3n}{2}-3}(n-2)^{\frac{3n}{2}+2}.$$

This proves that $\kappa(\Gamma_G)$ divides $\kappa(\Gamma_M)$. □

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